

TWO METHODS FOR THE LIGHT CURVES EXTREMA DETERMINATION

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Abstract: Two methods for the determination of extrema timings and their uncertainties appropriate for the analysis of time series of variable stars using matrix calculus are presented. The method I is suitable for determination of times of extrema of non-periodical variables or objects, whose light curves vary. The method II is apt for $O-C$ analyses of objects whose light curves are more or less repeating.

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1. Linear regression using matrix calculus

The determination of times of extrema and its uncertainty of light variations belongs to the most common problems in the astrophysics of variable stars. The following instructive text develops the basic ideas outlined in (1) Mikulášek et al. (2006).

1.1. The data

Let we have n observations $\{y_i\}$ creating the vector \mathbf{Y} , $\mathbf{Y} = \{y_i\}$ done in moments $\{t_i\}$ creating the vector \mathbf{t} , $\mathbf{t} = \{t_i\}$, both of the orders of $n \times 1$. It is demanding to adapt the time scale so that $\mathbf{t} \Rightarrow \mathbf{t} - \bar{\mathbf{t}}$.

The weight w_i expresses the quality of an i -th measurement; $\mathbf{w} = \{w_i\}$. If we know the inner uncertainty (error) of the i -th observation δy_i , we can determine the weight of it as follows: $w_i \sim (\delta y_i)^{-2}$. If we assume that the quality of all measurements are more or less equal or if we know about the quality of measurements little or nothing, it would be honest to put $w_i \equiv 1$, $\mathbf{w} = \mathbf{E}(n, 1)$.

The most of relations look simpler if we normalise weights so it is valid $\bar{w} = 1$, then $\sum_{i=1}^n w_i = n$. Frequently, we use instead of the \mathbf{w} square matrix $(n \times n)$ \mathbf{W} , $\mathbf{W} = \text{diag}(\mathbf{w})$.

1.2. Linear regression

The observed relationship between the dependent variable (inaccurately measured quantity - mostly magnitude, radial velocity, temperature) y and the independent variable (precisely measured quantity - typically time) t can be fit by an appropriate **model function** $F(t)$. The model function is determined by g free parameters β_j that create a column vector $\vec{\beta} = [\beta_1, \beta_2, \dots, \beta_g]^T$. The upper index T denotes transposing the matrix. If the model function $F(t)$ can be expressed as a linear combination of g different functions of time $f_k(t)$ (we speak here about the linear model function). Then

$$\mathbf{f}(t) = [f_1(t), f_2(t), \dots, f_g(t)], \quad F(t, \vec{\beta}) = \sum_{k=1}^g \beta_k f_k(t) = \mathbf{f}(t) \vec{\beta}. \quad (1)$$

Let introduce the matrix \mathbf{X} of rank $n \times g$ and the column vector \mathbf{Y}_p , $(n \times 1)$:

$$\mathbf{X} = \begin{pmatrix} \mathbf{f}(t_1) \\ \mathbf{f}(t_2) \\ \dots \\ \mathbf{f}(t_n) \end{pmatrix}; \quad \mathbf{Y}_p = \begin{pmatrix} F(t_1) \\ F(t_2) \\ \dots \\ F(t_n) \end{pmatrix} = \mathbf{X} \vec{\beta}. \quad (2)$$

As the objective measure of the success rate of the fit for an a $\vec{\beta}$ is used usually the sum of weighted squares of deflection of observed values and predicted ones $S(\vec{\beta})$:

$$S(\vec{\beta}) = (\mathbf{Y} - \mathbf{Y}_p)^T \mathbf{W} (\mathbf{Y} - \mathbf{Y}_p) = \mathbf{Y}^T \mathbf{W} \mathbf{Y} - 2 \beta^T \mathbf{U} + \beta^T \mathbf{V} \beta, \quad (3)$$

where

$$\mathbf{U} = \mathbf{X}^T \mathbf{W} \mathbf{Y}; \quad \mathbf{V} = \mathbf{X}^T \mathbf{W} \mathbf{X}; \quad \mathbf{H} = \mathbf{V}^{-1}. \quad (4)$$

The least square method (LSM) considers the fit by the model function $F(t, \vec{\beta})$ as the best one if the sum $R = S(\vec{\beta} = \mathbf{b})$ is minimal. In the case of the linear model function $F(t, \vec{\beta})$ we obtain for \mathbf{b} and R :

$$\left. \frac{\partial S}{\partial \vec{\beta}} \right|_{\vec{\beta}=\mathbf{b}} = \mathbf{0} = -2 \mathbf{U} + 2 \mathbf{V} \mathbf{b}, \quad \Rightarrow \quad \mathbf{b} = \mathbf{H} \mathbf{U}; \quad R = \mathbf{Y}^T \mathbf{W} \mathbf{Y} - \mathbf{b}^T \mathbf{U}. \quad (5)$$

1.3. The example

Standardly used linear regression models are polynomials or trigonometric functions of arbitrary orders. As an example we select the parabolic model

- the simplest model we can use for fit of the real function in the form:
 $F(t) = \beta_1 t^2 + \beta_2 t + \beta_3$, $\mathbf{f}(t) = [t^2, t, 1]$, $\mathbf{X} = [\{t_i^2\} \{t_i\} \{1\}]$.

1.4. Standard deviation. Uncertainties

The standard deviation σ can be estimated using relation

$$\sigma = \sqrt{\frac{R}{n - g}}. \quad (6)$$

The components of the column vector $\delta\mathbf{b}$ used to be considered as a rigorous estimate of the uncertainty of the particular parameters. Unfortunately, they have this meaning only exceptionally, nevertheless it is sometimes required by referees. Contrary, very valuable is the following estimate of the uncertainty of the model predictions $\delta F(t)$

$$\delta\mathbf{b} = \sigma \sqrt{\text{diag}(\mathbf{H})}; \quad \delta F(t) = \sigma \sqrt{\mathbf{f}(t) \mathbf{H} \mathbf{f}(t)^T}. \quad (7)$$

2 Method I

Method I of the determination of the extrema times consists of several steps.

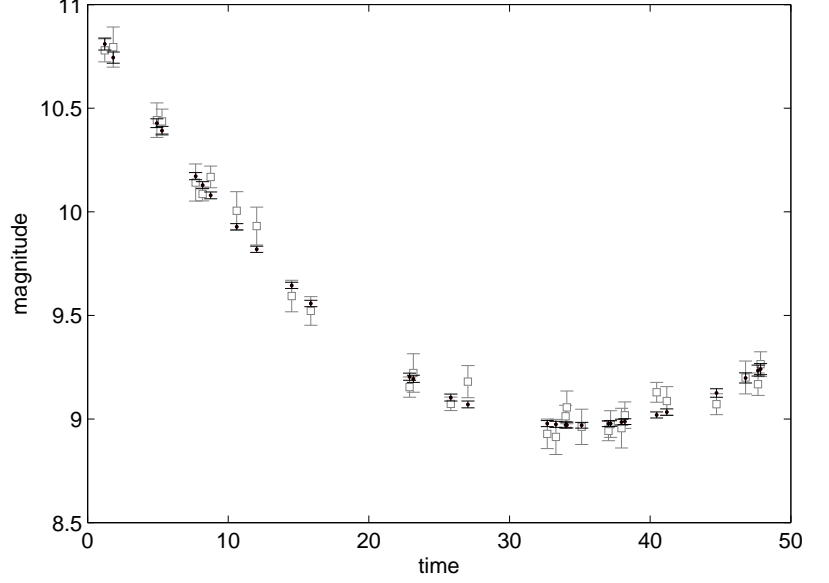
- Plot the time series and select the appropriate linear model function which can fit the observed light curve sufficiently well with the minimum of free parameters.
- Fit the observed light curve by the model function and check whether the dependence of deflections of observed and predicted light curves show only random scatter or some trends. In the latter case improve your model function
- Find the extreme time on the fitted light curve and calculate its uncertainty.

2.1. When the minimum/maximum occurs?

The moments of the fitted model function extrema occurs if it is fulfilled the condition that time derivative in that t_e equals to zero, especially

$$0 = \dot{y}_p(t_e) = \dot{F}(t_e) = \frac{d(\mathbf{f}(t_e) \mathbf{b})}{dt} = \dot{\mathbf{f}}(t_e) \mathbf{b}; \quad \dot{\mathbf{f}}(t) = [\dot{f}_1(t), \dot{f}_2(t), \dots, \dot{f}_g(t)]. \quad (8)$$

There are many techniques how to find roots of the derivative of the fitted function $\dot{F}(t)$. The uncertainty of the determination of the time derivative



Obrázek 1: The simulated light curve of a variable star with a minimum. Inner accuracy of individual measurements are denoted by gray error bars. The fitted parabola is marked by black dots with error bars corresponding to the uncertainty of predicted values.

at the time t , $\delta\dot{y}_p(t)$ can be estimated by the relation:

$$\delta\dot{y}_p(t) = \sigma \sqrt{\dot{\mathbf{f}}(t) \mathbf{H} \dot{\mathbf{f}}(t)^T}. \quad (9)$$

For the estimate of the uncertainty of the time of extreme δt_e we need yet the second time derivative in of the fitted curve in the time of extreme $\ddot{y}_p(t_e)$.

$$\ddot{y}_p(t_e) = \ddot{\mathbf{f}}(t_e) \mathbf{b}; \quad \ddot{\mathbf{f}}(t) = [\ddot{f}_1(t), \ddot{f}_2(t), \dots, \ddot{f}_g(t)]; \quad \delta t_e = \frac{\delta\dot{y}_p(t_e)}{|\ddot{y}_p(t_e)|}. \quad (10)$$

3 Method II

This method is again an application of the weighted least square method this once with the model function defined so that time or times of extrema are free parameters which are found and iteratively improved. Uncertainties

of these parameters are then simply uncertainties of determined extrema moments.

The method is applied namely for periodic or nearly periodic variable stars as pulsating stars, eclipsing binaries or rotating stars. The disadvantage of the method consists in the fact that from principle reasons we cannot applied here linear regression as we have to used approaches established for solution of non-linear LSM.

3.1. Example - transformation of the parabolic model

In section 1.3 we introduced the linear model expression with three parameters $\vec{\beta}$, $F(t) = \beta_1 t^2 + \beta_2 t + \beta_3$. This parabolic model function can be rewritten in the non-linear form:

$$F(t, \vec{\alpha}) = \alpha_2 (t - \alpha_1)^2 + \alpha_3, \quad (11)$$

where the parameter $\alpha_1 = t_{\min}$ corresponds to the time of the minimum of quadratic model function, $\alpha_1 = -\beta_2/2\beta_1$, $\alpha_2 = \beta_1$, $\alpha_3 = \beta_3 - \beta_2^2/4\beta_1$. It is apparent that the model $F(t, \vec{\alpha})$ as it is expressed in Eq. 11 is no more linear, nevertheless it is in the form demanded by the method II.

3.2. Linearisation of non-linear model functions

The solution of non linear regression is not straightforward and immediate than in the case of linear regression. Nevertheless, having a satisfactorily an initial estimate of the solution \vec{a}_0 we can linearise the non-linear model function and delight in all advantages yielding by linear models. The linearisation can be done by the Taylor decomposition of the first order

$$F(t, \vec{a}_1) \cong F(t, \vec{a}_0) + \sum_{j=1}^g \Delta a_j \frac{\partial F}{\partial \alpha_j} \quad (12)$$

The function is now linear in respect to new set parameters $\Delta \mathbf{a}$, functions of the linearised model are $\mathbf{f}(t) = [\frac{\partial F(t)}{\partial \alpha_1}, \frac{\partial F(t)}{\partial \alpha_2}, \dots, \frac{\partial F(t)}{\partial \alpha_g}]$.

Now we can create matrices \mathbf{X} (see Eq. 2), \mathbf{Y} , \mathbf{W} and calculate a solution for the vector $\Delta \mathbf{a}$.

$$\mathbf{V} = \mathbf{X}^T \mathbf{W} \mathbf{X}; \quad \mathbf{U} = \mathbf{X}^T \mathbf{W} \mathbf{Y}; \quad \mathbf{H} = \mathbf{V}^{-1}; \quad \Delta \mathbf{a} = \mathbf{H} \mathbf{U}. \quad (13)$$

This correction we add to the initial estimate \mathbf{a}_0 and we obtained the further, improved estimate for the vector of parameters $\mathbf{a}_1 = \mathbf{a}_0 + \Delta \mathbf{a}$. The new value of parameter set we can repeat the whole procedure. The iterative process diminish the value of the correcting vector $\Delta \mathbf{a}$ as rule very effectively and after several steps we obtain the final result.

You need not to iterate all parameters, some of them (the linear ones) can be calculate directly - see the following example.

3.3. Example - linearisation of the parabolic model

Now we can linearise our quadratic model according to the Eq. 12, assuming the initial estimate of parameters \mathbf{a}_0

$$F = [a_{02} (t - a_{01})^2 + a_{03}] + \Delta a_1 2 a_{02} (t - a_{01}) + \Delta a_2 (t - a_{01})^2 + \Delta a_3. \quad (14)$$

The function is in parameters a_2, a_3 linear. It means we can calculate the first two parameters directly, and a_1 iteratively:

$$F(t, \mathbf{a}) = \Delta a_1 2 a_2 (t - a_{01}) + a_2 (t - a_{01})^2 + a_3. \quad (15)$$

$$\mathbf{X} = \begin{pmatrix} 2 a_2 (t_1 - a_{01}) & (t_1 - a_{01})^2 & 1 \\ 2 a_2 (t_2 - a_{01}) & (t_2 - a_{01})^2 & 1 \\ \dots & \dots & \dots \\ 2 a_2 (t_n - a_{01}) & (t_n - a_{01})^2 & 1 \end{pmatrix} \quad (16)$$

Knowing parameters a_2, a_3 (repeating light variations) we should fix them.

3.4. Estimation of the uncertainty of extrema timings

The uncertainties of free parameters including the extremum timing are given by

$$\Delta \mathbf{Y} = \mathbf{Y} - \mathbf{F}(\mathbf{t}, \mathbf{a}); \quad R = \Delta \mathbf{Y}^T \mathbf{W} \Delta \mathbf{Y}; \quad \delta \mathbf{a} = \sigma \sqrt{\text{diag}(\mathbf{H})}. \quad (17)$$

Anyhow the resulting sets of correcting parameters are practically 'pure zero', their uncertainties differ from zero a correspond to uncertainties of the particular parameters. It enables to determine the reliable estimate for errors in extrema time determinations.

4 Discussion and conclusion

It can be also proven that in the case of identical model the both methods are interchangeable and yield the identical results.

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References

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